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APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

FIRST SEMESTER B.TECH DEGREE EXAMINATION(2019 SCHEME), DECEMBER 2019

Course Code: MAT101

Course Name: LINEAR ALGEBRA AND CALCULUS

(2019-Scheme)

Max. Marks: 100 Duration: 3 Hours

PART A

Answer all questions, each carries 3 marks.

1 Determine the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$ (3)

2 If 2 is an eigen value of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, without using its characteristic equation, find the other eigen values. (3)

3 If $f(x,y) = xe^{-y} + 5y$ find the slope of f (x,y) in the x-direction at (4,0). (3)

4 Show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$, where $z = e^x \sin y + e^y \cos x$ (3)

5 Find the mass of the square lamina with vertices (0,0) (1,0) (1,1) and (0,1) and density function $x^2 y$ (3)

6 Evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dxdy$ by changing to polar coordinates. (3)

7 Test the convergence of the series $\sum_{k=1}^{\infty} \dfrac{k}{2k+1}$ (3)

8 Check the convergence of $\sum_{k=1}^{\infty} \dfrac{1}{k^{k/2}}$ (3)

9 Find the Taylors series for $f(x) = \cos x$ about $x = \dfrac{\pi}{2}$ up to third degree terms. (3)

10 Find the Fourier half range sine series of $f(x) = e^x$ in $0 < x < 1$ (3)

PART B

Answer one full question from each module, each question carries 14 marks

Module-I

11 a) Solve the system of equations by Gauss elimination method. (7)

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

b) Find the eigenvalues and eigenvectors of (7)

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

12 a) Find the values of λ and μ for which the system of equations (7)

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \lambda z = \mu$$

has (i) no solution (ii) a unique solution and (iii) infinite solution

b) Find the matrix of transformation that diagonalize the matrix (7)

$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$. Also write the diagonal matrix.

Module-II

13 a) Let f be a differentiable function of three variables and suppose that $w = f(x-y, y-z, z-x)$,show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$ (7)

b) Locate all relative extrema of $f(x,y) = 4xy - y^4 - x^4$ (7)

14 a) Find the local linear approximation L to the function $f(x,y) = \sqrt{x^2 + y^2}$ at the point P(3,4).Compare the error in approximating f by L at the point Q(3.04,3.98) with the distance PQ. (7)

b) The radius and height of a right circular cone are measured with errors of at most 1% and 4%, respectively. Use differentials to approximate the maximum percentage error in the calculated volume. (7)

Module-III

15 a) Evaluate $\iint_R y\,dx\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$. (7)

b) Use double integral to find the area of the region enclosed between the parabola $y=\dfrac{x^2}{2}$ and the line $y=2x$. (7)

16 a) Evaluate $\int_0^2 \int_{\frac{y}{2}}^{1} e^{x^2}\,dx\,dy$ by reversing the order of integration (7)

b) Use triple integrals to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z=1$ and $x + z = 5$. (7)

Module-IV

17 a) Find the general term of the series $1+\dfrac{1\cdot 2}{1\cdot 3}+\dfrac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\dfrac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\ldots\ldots$and use the ratio test to show that the series converges. (7)

b) Test whether the following series is absolutely convergent or conditionally convergent $\sum_{k=1}^{\infty}\dfrac{(-1)^k}{\sqrt{k(k+1)}}$ (7)

18 a) Test the convergence of $\dfrac{x}{1.2}+\dfrac{x^2}{2.3}+\dfrac{x^3}{3.4}+\ldots\ldots\ldots\dfrac{x^k}{k(k+1)}+\text{--------}$ (7)

b) Test the convergence of the series $\sum_{k=1}^{\infty}\dfrac{(k+1)!}{4!\,k!\,4^k}$ (7)

Module-V

19 a) Find the Fourier series of periodic function with period 2 which is given below $f(x)=\begin{cases}-x\,; -1\le x\le 0\\ x\,;\ 0\le x\le 1\end{cases}$. Hence prove that $1+\dfrac{1}{3^2}+\dfrac{1}{5^2}+---=\dfrac{\pi^2}{8}$ (7)

b) Find the half range cosine series for $f(x) = \begin{cases}kx & 0\le x\le L/2\\ k(L-x) & L/2\le x\le L\end{cases}$ (7)

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a) Find the Fourier series of $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x^2 & 0 < x < \pi \end{cases}$ (7)

b) Obtain the Fourier series expansion for ` $f(x) = x^2, \quad -\pi < x < \pi$. (7)

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1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 5 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\cancel{R_3 \to R_3 - R_1}$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

$R_3 \to R_3 - R_1$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ rank of $A = \underline{\underline{3}}$

2. Let λ_2 and λ_3 be other eigen values.

Sum of eigen values = Sum of diagonal elements

ie $2 + \lambda_2 + \lambda_3 = 11$ —— ①

Product of eigen values = Determinant of matrix.

ie $2\lambda_2\lambda_3 = 36$ —— ②

From ① & ②, $\lambda_2 = 3$

$\lambda_3 = 6$

3. $f(x,y) = xe^{-y} + 5y$

Slope in x-direction $\Rightarrow \frac{d}{dx} f(x,y) = f_x$

$f_x = e^{-y}$

Slope at $(4,0) \Rightarrow$ substitute $x=4$, $y=0$ for f_x.

$\Rightarrow f_x = 1$ //

4. $z = e^x \sin y + e^y \cos x$

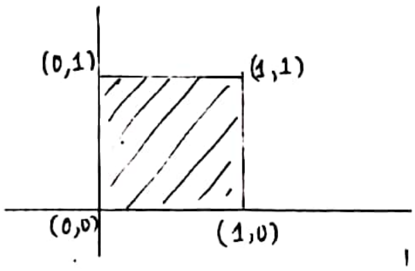
$\frac{\partial z}{\partial x} = e^x \sin y - e^y \sin x.$ $\qquad \frac{\partial z}{\partial y} = e^x \cos y + e^y \cos x$

$\frac{\partial^2 z}{\partial x^2} = e^x \sin y - e^y \cos x$ — ① $\qquad \frac{\partial^2 z}{\partial y^2} = -e^x \sin y + e^y \cos x$ — ②

① + ② $\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ //

5. Mass of square lamina given by $m = \iint \delta(x,y)\, dx\, dy$

where $\delta(x,y)$ = density function



$x : 0 \to 1$

$y : 0 \to 1.$

$\delta(x,y) = x^2 y$

$m = \int_0^1 \int_0^1 x^2 y \, dx\, dy$

$= \int_0^1 \left[\frac{x^3}{3} y\right]_0^1 dy$

$= \int_0^1 \frac{y}{3} dy = \left[\frac{y^2}{6}\right]_0^1 = \frac{1}{6}$

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6. $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ $\quad x : 0 \to \infty \quad y : 0 \to \infty \quad \Rightarrow \quad r : 0 \to \infty \quad \theta : 0 \to \pi/2$

$x^2+y^2 = r^2 \qquad dx\,dy = r\,dr\,d\theta$

$$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

$r^2 = t.$

$2r\,dr = dt.$

$$= \int_0^{\pi/2}\int_0^{\infty} \frac{e^{-t}}{2}\,dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} e^{-t}\Big]_0^{\infty} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} -\,d\theta$$

$$= \frac{1}{2} \times \theta\Big]_0^{\pi/2} = \underline{\underline{\pi/4}}$$

7. $\lim_{k\to\infty} U_k = \lim_{k\to\infty} \frac{k}{2k+1}$

$$= \lim_{k\to\infty} \frac{k}{k(2+1/k)} = 1/2 \neq 0$$

$\lim_{k\to\infty} U_k \neq 0 \Rightarrow$ Series is $\underline{\underline{\text{divergent}}}$

8. $\lim_{k\to\infty} (U_k)^{1/k} = \lim_{k\to\infty} \frac{1}{k^{1/2}}$

$$= 0$$

$\lim_{k\to\infty} (U_k)^{1/k} < 1$, according to root test

series is $\underline{\underline{\text{convergent}}}$.

9 $f(x) = \cos x$ $\quad f(\pi/2) = \cos \pi/2 = 0$

$f'(x) = -\sin x$ $\quad f'(\pi/2) = -\sin \pi/2 = -1$

$f''(x) = -\cos x$ $\quad f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(x) = \sin x$ $\quad f'''(\pi/2) = \sin \pi/2 = 1$

... ...

Taylor series expansion is given by

$$f(x) = f(a_0) + f'(a_0)\frac{(x-a_0)}{1!} + f''(a_0)\frac{(x-a_0)^2}{2!} + f'''(a_0)\frac{(x-a_0)^3}{3!} + \cdots$$

$$\cos x = 0 + -1 \times \frac{(x-\pi/2)}{1!} + 0 + 1 \times \frac{(x-\pi/2)^3}{3!} + \cdots$$

10. Half range sine series of $f(x)$, ~~...~~ $[0,L]$

given as $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$

$$b_n = \frac{2}{\pi}\int_0^1 e^x \sin n\pi x \, dx.$$

from std. soln

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$= \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^x}{1+n^2}\right.($$

$$= \frac{2n\pi}{1+n^2\pi^2}\left(1-(-1)^n e\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}\left(1-(-1)^n e\right)\sin n\pi x.$$

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PART - B

Module - I.

11. a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 2 & 2 \\ 3 & 3 & 4 & 1 \end{bmatrix}$

$R_1 \rightarrow R_2 - 2R_1.$

$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 3 & 3 & 4 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - 3R_1$

$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & -3 & -5 & -2 \end{bmatrix}$

$R_3 \rightarrow R_3 - 3R_2.$

$[A:B] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 7 & -2 \end{bmatrix}$ $\Rightarrow$ $x + 2y + 3z = 1$, $-y - 4z = 0$, $7z = -2.$ $\Rightarrow$ $x = -3/7$, $y = 8/7$, $z = -2/7$

b) $[A - \lambda I] = \begin{bmatrix} 4-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix}$

$|A - \lambda I| = 0.$

$\Rightarrow \lambda^3 - 12\lambda^2 + 39\lambda - 28 = 0$

$\lambda = 1, 4, 7$

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$\underline{\lambda = 1}$

$$[A - I] = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 4 & 0 \\ -2 & 0 & 2 \end{bmatrix} \qquad [A - \lambda I][X] = 0.$$

↳ Eigen vector.

$\cancel{3x_1 + 2x_2 - 2x_3 = 0}.$

$\cancel{2x_1 + 4x_2 + 0 = 0}$

$$\frac{x_1}{\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ -2 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 4 \\ -2 & 0 \end{vmatrix}}$$

$$k = \frac{x_1}{8} = \frac{-x_2}{4} = \frac{x_3}{8} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} 8 \\ -4 \\ 8 \end{bmatrix} = k\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$\underline{\lambda = 4}$

$$[A - 4I] = \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}}$$

$$k = \frac{x_1}{-1} = \frac{-x_2}{-2} = \frac{x_3}{2}. \qquad X = k\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$\underline{\lambda = 7}$

$$[A - 7I] = \begin{bmatrix} -3 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & -4 \end{bmatrix} \qquad \frac{x_1}{\begin{vmatrix} -2 & 0 \\ 0 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 0 \\ -2 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & -2 \\ -2 & 0 \end{vmatrix}}$$

$$k = \frac{x_1}{8} = \frac{-x_2}{-8} = \frac{x_3}{-4} \qquad X = k\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

12. a) $[A:B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

Refer. Q. 11.a) ls

→ Find Augmented matrix from the relation $AX = B$

→ Reducing augmented matrix.

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix}$$

i) When $\lambda = 5$ and $\mu \neq 9$ $\Rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & 0 & k \end{bmatrix}$

rank of $A \neq$ rank of $[A:B]$

∴ No solution

ii) When $\lambda \neq 5$ and ~~$\mu \neq 9$~~ μ = any value. $\Rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & a & b \end{bmatrix}$

rank of A = rank of $[A:B]$ = n (no. of variables)

∴ Unique solution.

iii) When $\lambda = 5$ and $\mu = 9$. $\Rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank of A = rank of $[A:B]$ $< n = 3$

∴ Infinite solutions.

b) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Refer Q. 11 .b) ls

→ Find Eigen values of a matrix by solving characteristic equation from the relation $|A - \lambda I| = 0$.

$\lambda^3 - 12\lambda - 16 = 0$

$\lambda = -2, -2, 4$

→ Find eigen vector using eigen value substituted matrix $[A - \lambda I]$

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Eigen vector are $\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix}$

matrix $P = [X_1 \ X_2 \ X_3]$

$$P = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\-1 & 0 & 2\end{bmatrix}$$

$P^{-1}AP = D$, diagonal matrix.

$$D = \begin{bmatrix}\lambda_1 & 0 & 0\\0 & \lambda_2 & 0\\0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & 4\end{bmatrix} //$$

Module - II

13. a) put $r = x-y$ $s = y-z$ $t = z-x$.

$$w = f(x-y, y-z, z-x) = f(r,s,t)$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \times \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \times \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \times \frac{\partial t}{\partial x}.$$

$$= \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} \quad \text{—①}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \times \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \times \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \times \frac{\partial t}{\partial y}$$

$$= \frac{\partial w}{\partial s} - \frac{\partial w}{\partial r} \quad \text{—②}$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} - \frac{\partial w}{\partial s} \quad \text{—③}$$

$$① + ② + ③ \Rightarrow \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} + \frac{\partial w}{\partial s} - \frac{\partial w}{\partial r} + \frac{\partial w}{\partial t} - \frac{\partial w}{\partial s}$$

$$= \underline{\underline{0}}$$

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b) $f(x,y) = 4xy - y^4 - x^4$

$f_x = 4y - 4x^3$ $\quad$ $f_y = 4x - 4y^3$

$f_{xx} = -12x^2$ $\quad$ $f_{yy} = -12y^2$ $\quad$ $f_{xy} = 4$

$f_x = f_y = 0 \Rightarrow$ critical points.

$4y - 4x^3 = 0.$ $\quad$ $4x - 4y^3 = 0$

$4y = 4x^3$ $\quad$ $4x = 4y^3$

$y = x^3$ $\quad$ $x = y^3.$

$y = (y^3)^3$

$y = 1, 0, -1$

$x = 1, 0, -1$

Critical points are $(1,1), (0,0), (-1,-1)$

$D = f_{xx} * f_{yy} - (f_{xy})^2$

For $\underline{(0,0)}$

$f_{xx} = 0$

$f_{yy} = 0$

$f_{xy} = 4.$

$D = 0 - (4)^2$

$= -16 < 1.$

$(0,0)$ is saddle point

$\underline{(1,1)}$

$f_{xx} = -12$

$f_{yy} = -12$

$D = 144 - 16$

$= 128 > 0$

$f_{xx} = -12 < 0.$

$(1,1)$ is maxima

$\underline{(-1,-1)}$

$f_{xx} = -12$

$f_{yy} = -12$

$D = 144 - 16$

$= 128 > 0$

$f_{xx} = -12 < 0$

$(-1,-1)$ is maxima

14. a) $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$(x_0,y_0) = (3,4)$

$f(x,y) = \sqrt{x^2+y^2}$ $\quad$ $f(x_0,y_0) = \sqrt{9+16} = 5$

$f_x = \frac{1}{2\sqrt{x^2+y^2}} \times 2x$ $\quad$ $f_x(x_0,y_0) = \frac{3}{5}$

$f_y = \frac{1}{2\sqrt{x^2+y^2}} \times 2y$ $\quad$ $f_y(x_0,y_0) = \frac{4}{5}$

$L(x,y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$

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$L(Q)$: substitute $x = 3.04$ & $y = 3.98$ in $L(x, y)$.

$= 5.008$

$f(Q)$ = Substitute $x = 3.04$ & $y = 3.98$ in $f(x,y) = \sqrt{x^2+y^2}$

$= 5.00819$.

$L(Q) - f(Q) = -0.00019$

distance $\overline{PQ} = \sqrt{(3.04-3)^2 + (3.98-4)^2}$

$= 0.045$

$$\text{Error} = \frac{|L(Q) - f(Q)|}{\overline{PQ}} = 0.0042$$

b) Vol of cone $= \frac{1}{3}\pi r^2 h$, r = radius of cone
h = height of cone

$\frac{dr}{r} \times 100 = 1$ $\qquad$ $\frac{dh}{h} \times 100 = 4$

$V = \frac{1}{3}\pi r^2 h$.

$dV = \frac{\pi}{3}(r^2 dh + h 2r dr)$ $\rightarrow$ dividing by V

$$\frac{dV}{V} = \frac{\frac{\pi}{3}(r^2 dh + 2rh dr)}{\frac{1}{3}\pi r^2 h}$$

$$\frac{dV}{V} = \frac{dh}{h} + 2\frac{dr}{r}$$

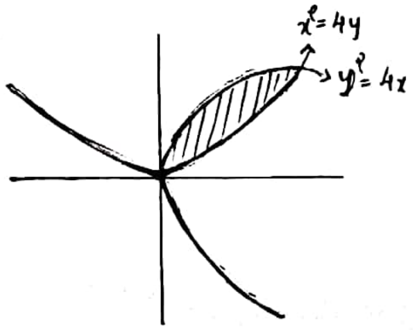
$= 0.04 + 2 \times 0.01$

$= 0.06$.

% error $= \underline{\underline{6\%}}$.

Module - III

15. a)



$x^2/4 = y$.

$x^4/16 = 4x \qquad x(x^3 - 4 \times 16) = 0$

$x = 0. \quad y = 0$

$x^3 = 64$

$x = 4 \quad y = 4$

Region $\Rightarrow$ with critical points (0,0) and (4,4) b/w the 2 parabolas.

$$\iint_R y\,dx\,dy \qquad y: 0 \to 4 \qquad x: y^2/4 \to 2\sqrt{y}$$

$$\Rightarrow \int_0^4 \int_{y^2/4}^{2\sqrt{y}} y\,dx\,dy$$

↳ Alternatively, we $x: 0 \to 4$, $y: x^2/4 \to 2\sqrt{x}$.

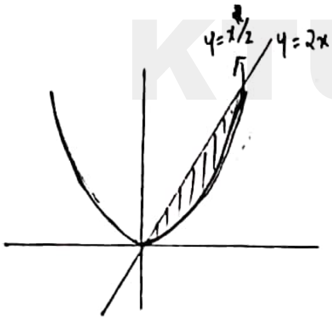
$$= \int_0^4 \left. y\,x \right]_{y^2/4}^{2\sqrt{y}} dy$$

$$= \int_0^4 y\left(2\sqrt{y} - y^2/4\right) dy.$$

$$= \left. \frac{2y^{3/2}}{3/2} - \frac{y^4}{16} \right]_0^4$$

$$= \frac{2 \times 4^{3/2}}{3/2} - \frac{4^4}{16} = \frac{48}{5}$$

b)



$x^2/2 = 2x.$

$x(x/2 - 2) = 0.$

$x = 0$, $y = 0$ $\qquad x/2 = 2.$ $\qquad x = 4. \quad y = 8$

Area b/w region (0,0) and (4,8) bounded by parabola & line.

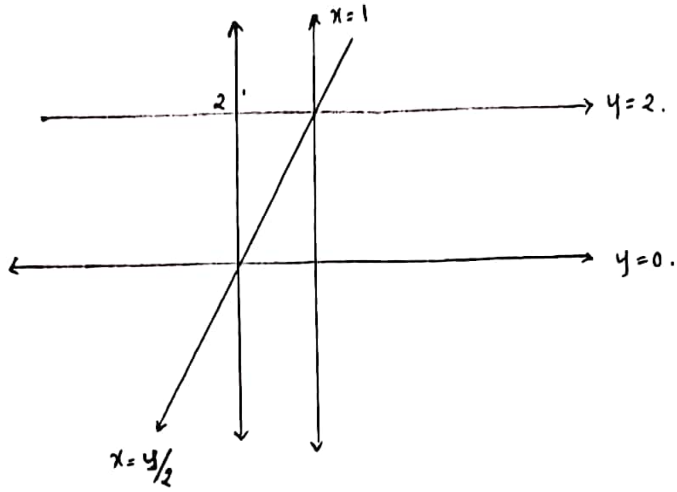
$$= \iint_R dx\,dy \qquad x: 0 \to 4 \qquad y: x^2/2 \to 2x.$$

$$= \int_0^4 \int_{x^2/2}^{2x} dy\,dx.$$

$$= \int_0^4 (2x - x^2/2)\,dx$$

$$= \left. x^2 - \frac{x^3}{6} \right]_0^4 = \frac{16}{3}$$

16. a) $$\int_{y=0}^{2}\int_{x=y/2}^{1} e^{x^2}\,dx\,dy$$



$y : 0 \to 2.$

$x : y/2 \to 1.$

$\Rightarrow x : 0 \to 1$

$y : 0 \to 2x.$

Integral $\Rightarrow$ $$\int_0^1\int_0^{2x} e^{x^2}\,dy\,dx.$$

$$= \int_0^1 e^{x^2} y\Big]_0^{2x} dx.$$

$$= \int_0^1 e^{x^2} \times 2x\,dx.$$

$$= \underline{\underline{e-1}}$$

b) Volume $= \iiint_G dV = \iint_R \int_{z=1}^{5-x} dz\,dy\,dx$ $\quad z : 1 \to 5-x.$

$$= \iint_R (5-x-1)\,dy\,dx.$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)\,dy\,dx.$$

$$= \int_{-3}^{3} (4-x)\times 2\sqrt{9-x^2}\,dx.$$

$$= \underline{\underline{36\pi}}$$

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17 (a) Series $\Rightarrow 1 + \frac{1\cdot 2}{1\cdot 3} + \frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5} + \frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7} + \cdots\cdots$

$\Rightarrow$ series $= \sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\cdot \cdots\cdots n}{1\cdot 3\cdot 5\cdot \cdots\cdots (2n-1)}$

$\therefore$ General term $\rightarrow a_n = \frac{1\cdot 2\cdot 3\cdots\cdots n}{1\cdot 3\cdot 5\cdots\cdots (2n-1)}$

$$a_{n+1} = \frac{1\cdot 2\cdot 3\cdot \cdots\cdot n(n+1)}{1\cdot 3\cdot 5\cdots (2n-1)(2n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1\cdot 2\cdot 3\cdot \cdots\cdot n\,(n+1)}{1\cdot 3\cdot 5\cdot \cdots (2n-1)(2n+1)}}{\frac{1\cdot 2\cdot 3\cdot \cdots\cdots n}{1\cdot 3\cdot 5\cdots\cdots (2n-1)}} = \frac{n+1}{2n+1} = \frac{n(1+1/n)}{n(2+1/n)}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(1+1/n)}{(2+1/n)}$$

Applying ratio test.

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(1+1/n)}{(2+1/n)} = \frac{1}{2} = l.$$

$l < 1$

$\therefore$ Series convergent by ratio test.

(b) Series $\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k(k+1)}} \Rightarrow a_k = \frac{(-1)^k}{\sqrt{k(k+1)}}$

$$|a_k| = \frac{1}{\sqrt{k(k+1)}}$$

$$|a_{k+1}| = \frac{1}{\sqrt{(k+1)(k+2)}}$$

$|a_{k+1}| < |a_k|$

$$\lim_{k\to\infty} |a_k| = \lim_{k\to\infty} \frac{1}{\sqrt{k(k+1)}} = 0$$

$\therefore$ The series is absolutely convergent by Leibnitz's test. //

18) a) Series $\Rightarrow \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \dots \frac{x^k}{k(k+1)}$

Series $= \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

$a_n = \frac{x^n}{n(n+1)}$

$(a_n)^{1/n} = \left(\frac{x^n}{n(n+1)}\right)^{1/n} = \frac{x}{(n(n+1))^{1/n}}$

$(a_n)^{1/n} = \frac{x}{(n^2(1+1/n))^{1/n}}$

Applying root test

$\Rightarrow \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{x}{(n^2(1+1/n))^{1/n}} = \lim_{n\to\infty} \frac{x}{(n^2)^{1/n}}$

$\Rightarrow \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{x}{(n)^{2/n}} = \lim_{n\to\infty} \frac{x}{n^0} = \lim_{n\to\infty} x$

$= \underline{\underline{x}}$

$\therefore$ The Series is Convergent if $x < 1$

The Series is divergent if $x > 1$

when $x = 1$,

Series $\Rightarrow \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots \frac{1}{k(k+1)}$

Series $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \Rightarrow a_n = \frac{1}{n(n+1)}$

$a_{n+1} = \frac{1}{(n+1)(n+2)}$ and $a_n = \frac{1}{n(n+1)}$

$\Longrightarrow a_n > a_{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = \underline{\underline{0}}$$

$\therefore$ The series is $\underline{\underline{\text{convergent}}}$ by Liebnit's test

b) $$\text{Series} = \sum_{k=1}^{\infty} \frac{(k+1)!}{4!.k!.4^k} = \sum_{k=1}^{\infty} \frac{k!(k+1)}{4!.k!.4^k}$$

$$\Rightarrow \text{series} = \sum_{k=1}^{\infty} \frac{(k+1)}{4!.\times 4^k} \Rightarrow a_k = \underline{\underline{\frac{k+1}{4!.\times 4^k}}}$$

$$a_{k+1} = \frac{k+2}{4!\times 4^{k+1}}$$

$$\Rightarrow \frac{a_{k+1}}{a_k} = \frac{k+2}{4!\times 4^k \times 4} \times \frac{4!\times 4^k}{(k+1)}$$

$$\Rightarrow \frac{a_{k+1}}{a_k} = \frac{k(1+2/k)}{4\times k(1+1/k)} = \frac{(1+2/k)}{4(1+1/k)}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1+2/k}{4(1+1/k)} = \frac{1}{4} = l$$

$$l < 1$$

By ratio test the series is $\underline{\underline{\text{convergent}}}$

17) a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

$2L = 2$

$L = 1$

$a_0 = \frac{1}{L}\int_{-1}^{1} f(x)\,dx$

$= \frac{1}{1}\left[\int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx\right]$

$= \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = 1/2 + 1/2$

$= 1$

$a_n = \frac{1}{L}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{L}\,dx$

$= 1/1 \int_{-1}^{0} -x\cos n\pi x\,dx + \int_0^1 x\cos n\pi x\,dx$

$= \left[-x\frac{\sin n\pi x}{n\pi}\right]_{-1}^{0} + \int_{-1}^{0}\frac{\sin n\pi x}{n\pi}dx + \left[x\frac{\sin n\pi x}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi}dx$

$= 0 + \left[-\frac{\cos n\pi x}{n^2\pi^2}\right]_{-1}^{0} + 0 - \left[-\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1$

$= \frac{-1+(-1)^n}{n^2\pi^2} - \left(\frac{-(-1)^n+1}{n^2\pi^2}\right)$

$= \frac{-2+2(-1)^n}{n^2\pi^2}$

$b_n = \frac{1}{L}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{L}\,dx$

$= 1/1 \int_{-1}^{0} -x\sin n\pi x\,dx + \int_0^1 x\sin n\pi x\,dx$

$= \left[x\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} - \int_{-1}^{0}\frac{\cos n\pi x}{n\pi}dx + \left[-x\frac{\cos n\pi x}{n\pi}\right]_0^1$

$$+\int_0^1 \cos\frac{n\pi x}{n\pi}\,dx$$

$$= \frac{(-1)^n}{n\pi} - \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_{-1}^{0} - \frac{(-1)^n}{n\pi} + \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \underline{0}$$

b) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= 2/L \int_0^{L/2} kx\,dx + \int_{L/2}^{L} k(L-x)\,dx$$

$$= \frac{2}{L}\left[\frac{kx^2}{2}\right]_0^{L/2} + \left[\frac{k(L-x)^2}{-2}\right]_{L/2}^{L}$$

$$= \frac{2}{L}\left[\frac{kL^2}{8} - 0 + 0 + \frac{kL^2}{8}\right]$$

$$= \frac{2}{L}\frac{kL^2}{4}$$

$$= \frac{kL}{2}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\left[\int_0^{L/2} kx\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} k(L-x)\cos\frac{n\pi x}{L}\,dx\right.$$

$$= \frac{2k}{L}\left[\frac{x\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]_0^{L/2} - \int_0^{L/2}\frac{\sin\frac{n\pi x}{L}\,dx}{n\pi/L} +$$

$$\left[\frac{(L-x)\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]_{L/2}^{L} + \int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2K}{L}\left[\frac{\frac{L}{2}\sin\frac{n\pi}{2}}{\frac{n\pi}{L}} - \left[\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right]_0^{L/2} - \frac{L}{2}\frac{\sin\frac{n\pi}{2}}{n\pi/L} + \left[\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right]_{L/2}^{L} \right]$$

$$= \frac{2K}{L}\left[\frac{\cos\frac{n\pi}{2} - 1 - (-1)^n + \cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{2K}{L} \times \frac{L^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$= \frac{2KL}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$\cancel{f(x) = \frac{KL}{4} + \sum_{n=1}^{\infty}\frac{2KL}{n^2\pi^2}\left(2\cos\frac{n\pi}{2}\right.}$$

$$f(x) = \frac{KL}{4} + \sum_{n=1}^{\infty}\frac{2KL}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right)\cos\frac{n\pi x}{L}$$

$$= \frac{KL}{4} + \frac{2KL}{\pi^2}\left[-\frac{4}{2^2}\cos\frac{2\pi x}{L} - \frac{4}{6^2}\cos\frac{6\pi x}{L} \cdots\right]$$

$$= \frac{KL}{4} - \frac{8KL}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \cdots\right]$$

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a) $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$$

$$2L = 2\pi$$

$$\Rightarrow \quad L = \pi$$

$$a_0 = \frac{1}{L} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} 0\, dx, \int_{0}^{\pi} x^2 dx \right\}$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi}$$

$$a_0 = \frac{\pi^3}{3\pi} = \underline{\underline{\frac{\pi^2}{3}}}$$

$$a_n = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi}{L} x\, dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} 0 \cos nx\, dx + \int_{0}^{\pi} x^2 \cos nx\, dx \right\}$$

$$= \frac{1}{\pi} \left\{ \left[x^2 \frac{\sin nx}{n} \right]_0^{\pi} - \int_{0}^{\pi} 2x \frac{\sin nx}{n} dx \right\}$$

$$= \frac{1}{\pi} \left\{ 0 - 2 \left(\left[\frac{-x \cos nx}{n^2} \right]_0^{\pi} + \int_{0}^{\pi} \frac{\cos nx}{n^2} dx \right) \right.$$

$$= \frac{-2}{\pi} \left\{ \frac{-\pi (-1)^n}{n^2} + \left[\frac{\sin nx}{n^3} \right]_0^{\pi} \right\}$$

$$a_n = \frac{2\pi (-1)^n}{\pi n^2} = \underline{\underline{\frac{2(-1)^n}{n^2}}}$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x^2 \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{2x\cos nx}{n}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} + \frac{2}{n}\left(\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} + \frac{2}{n}\left(0 - \left[-\frac{\cos nx}{n^2}\right]_0^{\pi}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} + \frac{2}{n}\left(\frac{-1+(-1)^n}{n^2}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}(-1+(-1)^n)\right\}$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{n^2}\cos nx + \frac{1}{\pi}\left\{\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}(-1+(-1)^n)\right\}\sin nx\right)$$

b) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

$2L = 2\pi$

$\Rightarrow L = \pi$

$$a_0 = \frac{1}{L}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \times \frac{2\pi^3}{3}$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{L} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left\{ \left[x^2 \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n} \, dx \right\}$$

$$= \frac{1}{\pi} \left\{ 0 - 2 \left(\left[-\frac{x \cos nx}{n^2} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n^2} \, dx \right) \right\}$$

$$= -\frac{2}{\pi} \left\{ \frac{-\pi(-1)^n - \pi(-1)^n}{n^2} + \left[\frac{\sin nx}{n^3} \right]_{-\pi}^{\pi} \right\}$$

$$= \frac{-2}{\pi} \times \frac{-2\pi(-1)^n}{n^2}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(x)\, \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$b_n = 0$$

$$\therefore f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x \cdots\right)$$